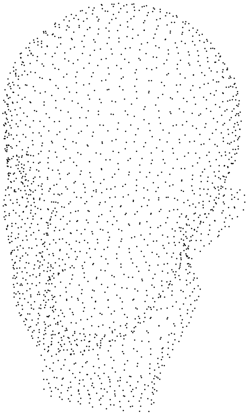
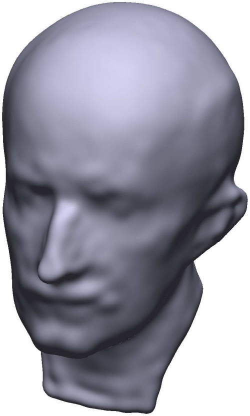
Minimizing Absolute Gaussian Curvature Locally

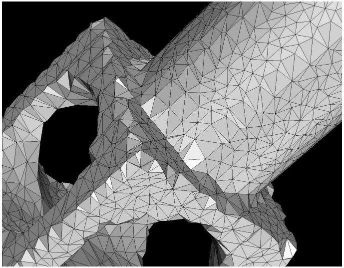
Joachim Giesen

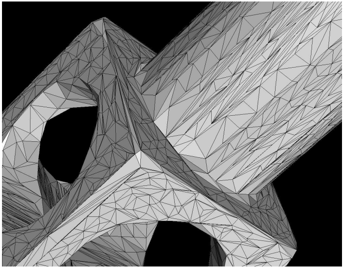
Surface Reconstruction





Mesh Smoothing





Tight Surfaces

- A surface is called tight if every hyperplane cuts the surface into at most two pieces (two piece property).
- Examples include convex polyhedra, torus of revolution.
- Generalization of convexity.

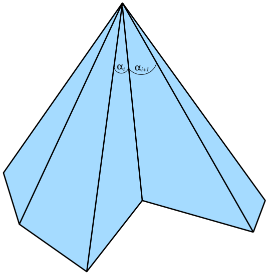
Absolute Gaussian Curvature

- Tight surface $\Leftrightarrow$ total absolute Gaussian curvature minimal.

Definitions

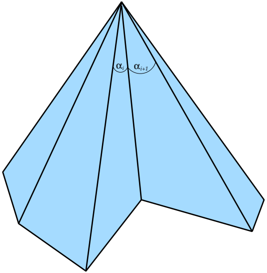
Gaussian Curvature

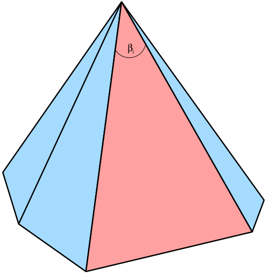
- For a smooth surface, the Gaussian curvature is the product of the principal curvatures.
- For a polyhedral surface, $K_v = 2\pi - \sum_i \alpha_i$.



Absolute Gaussian Curvature K_v^*:

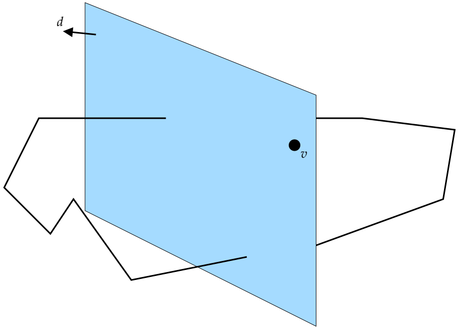
- For a smooth surface: absolute value of Gaussian curvature.
- For a polyhedral surface:
  - v in the convex hull of neighbors N_v
    $\Rightarrow K_v^* = \sum_i \alpha_i - 2\pi$.
  - v outside the convex hull of N_v
    $\Rightarrow K_v^* = 2\pi - 2\sum_j \beta_j + \sum_i \alpha_i$.





Integral Geometry

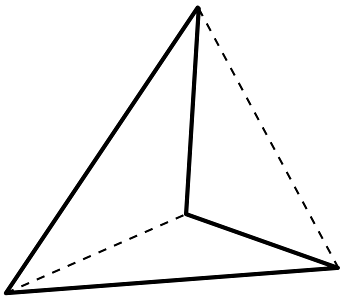
- $K_v^* = \frac{1}{2}\int_{d\in\mathbb{S}^2} |1 - i_{d,v}| do$
- Captures the "extent" of tightness of a surface.



Problem Statement

Given a polygon C in $\mathbb{R}^3$, find a point v in $\mathbb{R}^3/C$ such that the absolute Gaussian curvature of v with respect to C is minimized.

An Example



Non-constructiblity of an algebraic number

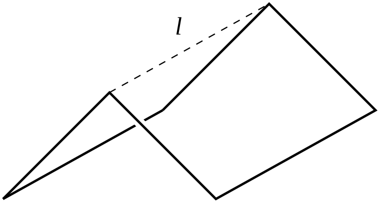
- Non constructiblity of an algebraic number $\Rightarrow$ cannot be expressed as finite sequence of $+, -, *, /$ and $k^{th} root$.
- Non constructibility $\Leftrightarrow$ minimal polynomial is unsolvable.
- Rational polynomial is unsolvable $\Leftrightarrow$ its Galois group is unsolvable
- S_n is unsolvable for $n \geq 5$.

Algebraic Hardness Result

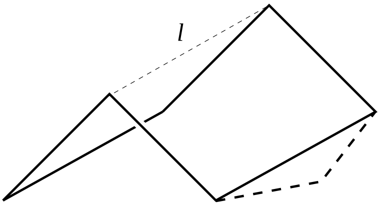
Theorem. The solution to the curvature minimization problem is in general not constructible.

In other words...
There exists a polygon C whose vertices have rational co-ordinates, but the co-ordinates of the unique solution are not constructible.

Construction of the Base Polygon



Construction of the Perturbed Polygon



Key Ideas of the Proof

- For sufficiently small perturbations of the base polygon the solution to the problem is unique and is in the interior of ℓ.
- The solution is the global minimum and hence also a local minimum on ℓ.
- The zero derivative condition of the local minimum can be converted into the vanishing of a rational polynomial.
- There are infinitely many of these rational polynomials with Galois group S_{10} as the perturbation tends to zero.

Some details of the Analysis

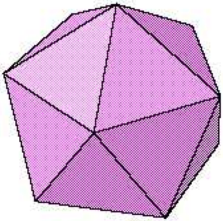
- For small perturbations of the base polygon the global minimum is still in the neigborhood of ℓ.
- For the base polygon, the gradient of K_v^* does not vanish in the neighbourhood of ℓ.
- For small perturbations of the base polygon this condition still holds!

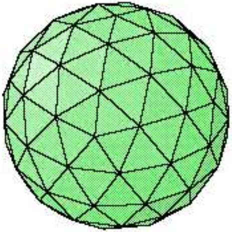
Approximating the value of the solution

- Algebraic hardness $\Rightarrow$ in general a solution cannot be computed exactly in the model of computation where roots of a polynomial are obtained from basic operations and k^{th} roots
- As an alternative, approximate the solution in value.

Approximation Algorithm

- Discretize S^2 by projecting the k^{th} order triangulation of the regular icosahedron.

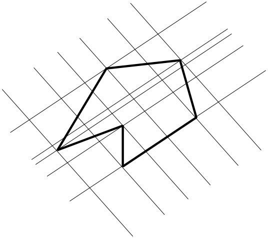




- Choose the circum centers of the triangles as the set of sampling directions D_k.

Approximation Algorithm

- This set of directions D_k induces a hyperplane arrangement in $\mathbb{R}^3$. Each cell in the arrangement has the same approximate curvature.



- Compute the approximate curvature $K_v^* = 1/2 \sum_{d \in D_k} w_k(d)|1 - i_{d,v}|$ for each cell, where $w_k(d)$ is the area of the spherical triangle for direction d.
- Pick a point in a cell with minimum value of approximate curvature.

Analysis

- For each $v \in \mathbb{R}^3$, we get a great circle arrangement C_v on the sphere such that all directions in a cell of this arrangement have the same index.
- We only make approximation errors when a spherical triangle associated with D_k intersects a great circle in C_v.
- For each spherical triangle, the error is bounded by $(n/2-1)w_k(d) \in O(n/k^2)$.
- Each great circle intersects at most $O(k)$ spherical triangles.
- Error of $O(n/k^2)$ can be made on $O(nk)$ triangles $\Rightarrow O(n^2/k)$ bound on the total error.

Open Problem

An efficient algorithm to approximate the solution in location.