# Finite Elements Methods for Wave Equations

**Workshop on Unstructured Meshes in Dynamical Spacetimes**

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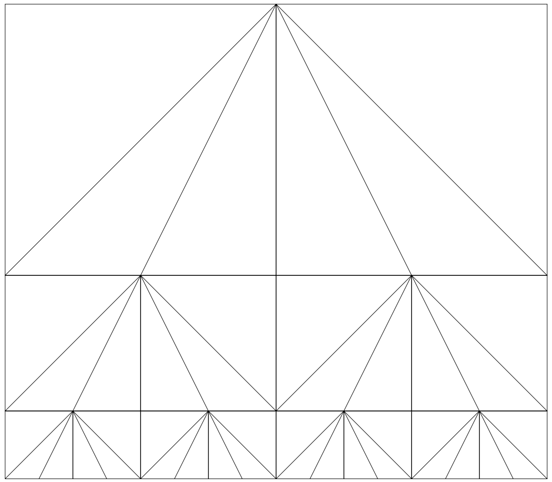
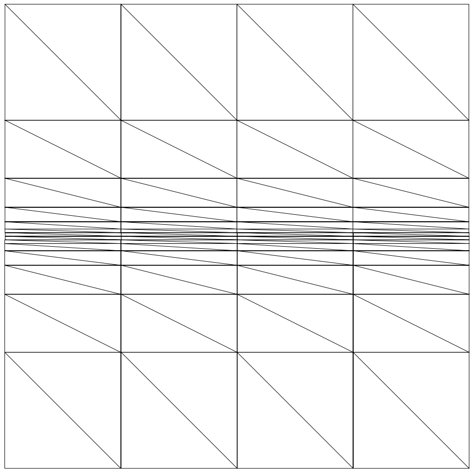
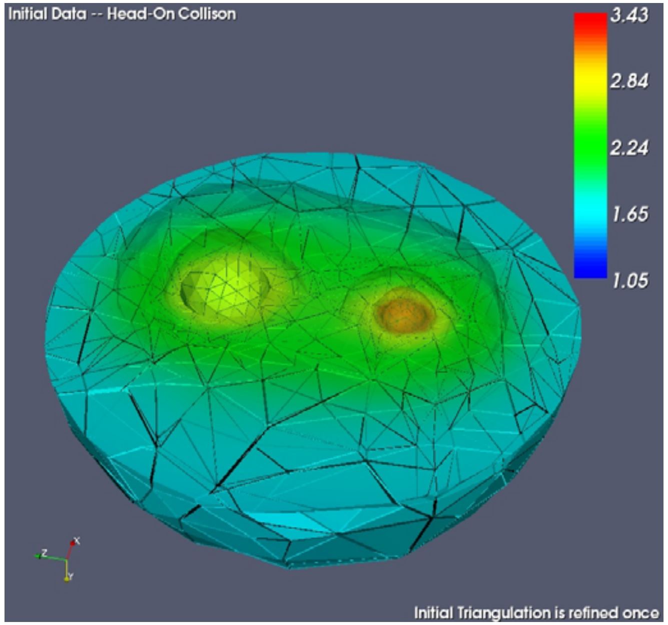
# Galerkin Method - Grids

Consistent discrete schemes for
- Unstructured grids
- Adaptive grid refinement
- Higher order approximation

- Spacetime grid
- Local time-stepping







# scalar, linear wave equation

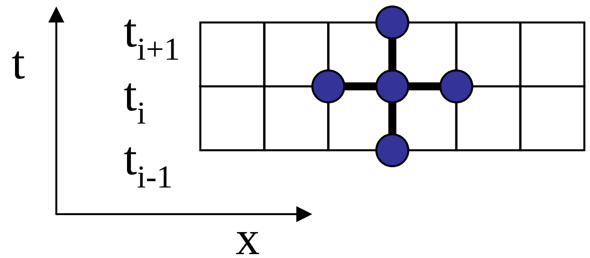
# Finite Differences in spacetime

$$\partial_{tt} u = \Delta u$$

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_0^2} - \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_1^2} = 0$$

$$u_{i+1} = 2u_i - u_{i-1} + (h_0)^2 \Delta_h u_i$$

leap-frog time stepping




CFL condition $h_0/h_k < 1$

# Galerkin Method – 1D

$$-u'' = f, \qquad u(0) = u(1) = 0$$

Strong formulation, **Finite Differences**

$$\Rightarrow \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i$$

$$\int_0^1 u' \cdot v'\, dx = \int_0^1 f \cdot v\, dx \qquad \forall v \in H_0^1(0,1)$$

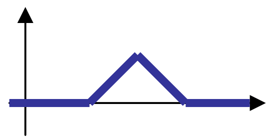
Weak formulation

$$\sum_i a_i \int_0^1 \phi_i' \cdot \phi_j'\, dx = \int_0^1 f \cdot \phi_j\, dx \qquad \forall j$$

Galerkin scheme

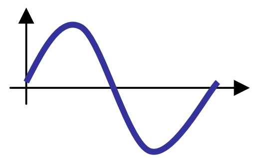
**Finite Element Method**

e.g. piecewise linear ϕ



**Pseudo-Spectral (Galerkin/ p) Method**

e.g. polynomial ϕ

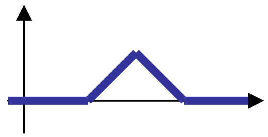


# FEM semidiscrete in space

$$\partial_{tt} u = \Delta u$$

$$\int_\Omega (\partial_{tt} u) w \, d^3x \quad = \quad - \int_\Omega (\nabla u) \cdot (\nabla w) d^3x \ \forall w$$

piecewise linear $\phi$ in space, continuous



$$M = \int_\Omega \phi_i \phi_j dx = h_1 [1 \ 4 \ 1]/6$$

$$A = \int_\Omega (\nabla \phi_i) \cdot (\nabla \phi_j) dx = \frac{1}{h_1} [-1 \ 2 \ -1]$$

$$M u_{tt} = -Au$$

~ FD scheme in space

solve system M

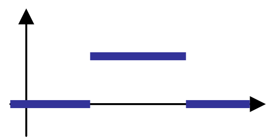
->choose numerical quadrature such that M is diagonal

# DG semidiscrete in time

$$\partial_t v = \Delta u$$
$$\partial_t u = v$$

$$\begin{aligned} -\int_T v(\partial_t w)dt &= \int_T (\Delta u) w \, dt \ \forall w \\ -\int_T u(\partial_t w)dt &= \int_T vw \, dt \ \forall w \end{aligned}$$

piecewise constant $\phi$ in time, discontinuous



$$A = \frac{1}{h_0}[-1 \ 1]$$

=Euler scheme in time

$$\begin{aligned} Av &= \Delta u^j \\ Au &= v^j \end{aligned}$$

# FEM in space, DG in time

$$\partial_t v = \Delta u$$
$$\partial_t u = v$$

$$\begin{aligned}
\int_{\Omega\times T} v(\partial_t w) dt\, d^3x &= \int_{\Omega\times T} (\nabla u)\cdot(\nabla w) dt\, d^3x \;\forall w \\
-\int_{\Omega\times T} u(\partial_t w) dt\, d^3x &= \int_{\Omega\times T} vw\, dt\, d^3x \;\forall w \;,
\end{aligned}$$

piecewise linear $\phi$ in space, continuous

piecewise constant $\phi$ in time, discontinuous

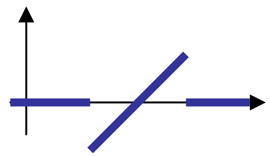
# interior penalty DG semidiscrete in space

$$\partial_{tt} u = \Delta u$$

$$\{u\} := ((u|_{\mathrm{element}_i}) + (u|_{\mathrm{element}_j}))/2$$

$$[u] := (u|_{\mathrm{element}_i}) - (u|_{\mathrm{element}_j})$$

piecewise linear ϕ in space, discontinuous



$$\int_\Omega (\partial_{tt} u) w \, d^3x \quad = \sum_i \int_{\mathrm{element}_i} (\nabla u) \cdot (\nabla w) d^3x$$
$$- \sum_{i<j} \int_{\mathrm{edge}_{ij}} \{n^{ij} \cdot \nabla u\}[w] d^2x$$
$$- \sum_{i<j} \int_{\mathrm{edge}_{ij}} [u]\{n^{ij} \cdot \nabla w\} d^2x$$
$$+ \sum_{i<j} \frac{c_p}{|\mathrm{edge}_{ij}|^{c_e}} \int_{\mathrm{edge}_{ij}} [u][w] d^2x \ \forall w$$

symmetric/
unsymmetric IPDG

Penalty parameters c

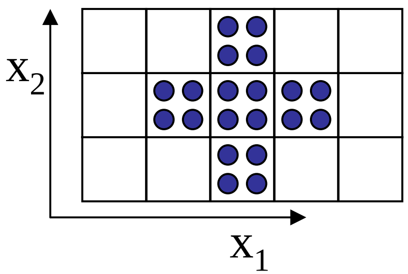
# interior penalty DG semidiscrete in space (2)

piecewise linear $\phi$ in space, discontinuous

$$(1-x_1)(1-x_2)$$
$$x_1(1-x_2)$$
$$(1-x_1)x_2$$
$$x_1x_2$$

$$M = \int_\Omega \phi_i \phi_j dx$$

$$Mu_{tt} = -Au$$




M block-diagonal
Solve system M
->choose $\phi$ such that M is diagonal

# Space-time Galerkin schemes

# New: FEM in spacetime

$$\int_{\Omega\times T}(\partial_t u)(\partial_t w)dt\, d^3x \quad = \quad \int_{\Omega\times T}(\nabla u)\cdot(\nabla w)\, dt\, d^3x\ \forall w$$

piecewise linear $\phi$ in spacetime, continuous

$$M = \int_\Omega \phi_i \phi_j dx = h_1[1\ 4\ 1]/6$$

$$A = \int_\Omega (\nabla \phi_i)\cdot(\nabla \phi_j)dx = \frac{1}{h_1}[-1\ 2\ -1]$$

$$\left(M - \frac{h_0^2}{6}A\right)u_{i+1} = \left(M + \frac{2h_0^2}{3}A\right)u_i - \left(M - \frac{h_0^2}{6}A\right)u_{i-1}$$

Solve system

# Spacetime as Petrov-Galerkin FEM

scalar wave equation

$$-\square u = 0$$

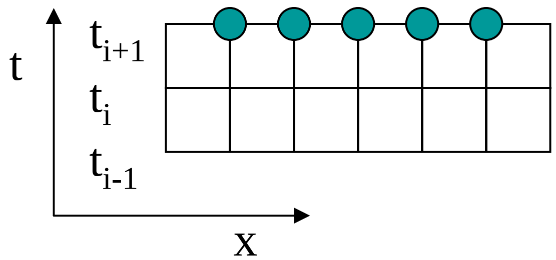
$$u(t=0) = u_0, \qquad u_t(t=0) = u_1$$

Petrov-Galerkin:
ansatz u & test v differ

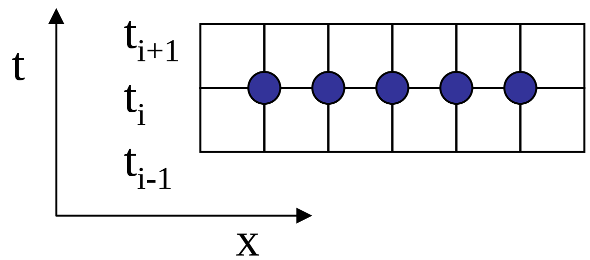
$$a(u,v) = \int_{\mathcal{M}} (-u_t v_t + \nabla u \cdot \nabla v)\mathrm{d}x = 0 \forall v$$

$$v(t=0) = 0, \qquad v(t=t_{\mathrm{ende}}) = 0$$

bi-linear ansatz function $u$




bi-linear test function $v$

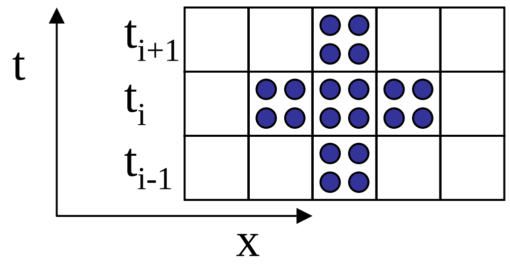



time stepping

# New: interior penalty DG in spacetime

$$\sum_i \int_{\mathrm{element}_i} \eta^{\alpha\beta}(\partial_\alpha u)(\partial_\beta w) d^4x$$
$$- \sum_{i<j} \int_{\mathrm{edge}_{ij}} \{\eta^{\alpha\beta} n^{ij}_\alpha \partial_\beta u\}[w] d^3x$$
$$- \sum_{i<j} \int_{\mathrm{edge}_{ij}} [u]\{\eta^{\alpha\beta} n^{ij}_\alpha \partial_\beta w\} d^3x$$
$$+ \sum_{i<j} \frac{c_p}{|\mathrm{edge}_{ij}|^{c_e}} \eta^{\alpha\beta} n^{ij}_\alpha n^{ij}_\beta \int_{\mathrm{edge}_{ij}} [u][w] d^3x = 0 \ \forall w$$

piecewise linear $\phi$ in spacetime, discontinuous



# New: interior penalty DG in spacetime(2)

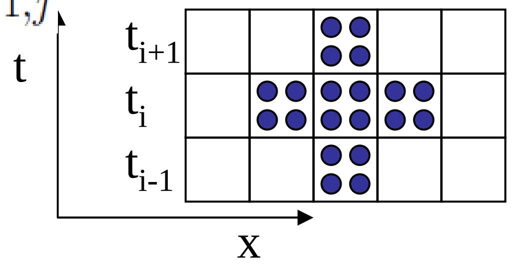
$(1-x_0)(1-x_1)$
$x_0(1-x_1)$
$(1-x_0)x_1$
$x_0x_1$

$$A_{1,0} = A^*_{-1,0} = \frac{h_1}{12h_0}\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -4 & -2 & 2 & 1 \\ -2 & -4 & 1 & 2 \end{pmatrix} + \frac{c_{p1}}{6}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$$

$$A_{0,-1} = A^*_{0,1} = \frac{h_0}{12h_1}\begin{pmatrix} 2 & -4 & 1 & -2 \\ 0 & 2 & 0 & 1 \\ 1 & -2 & 2 & -4 \\ 0 & 1 & 0 & 2 \end{pmatrix} + \frac{c_{p0}}{6}\begin{pmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{0,0} = -\frac{c_{p0}h_1}{6}\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} + \frac{c_{p1}h_0}{6}\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

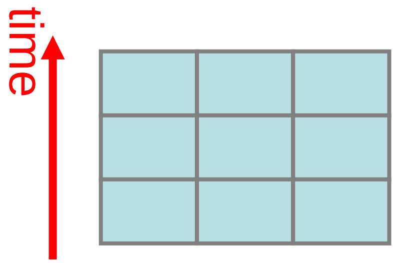
$$A_{1,0}u_{i+1,j} = A_{0,-1}u_{i,j-1} + A_{0,0}u_{i,j} + A_{0,1}u_{i,j+1} - A_{-1,0}u_{i-1,j}$$




Solve block-diagonal system

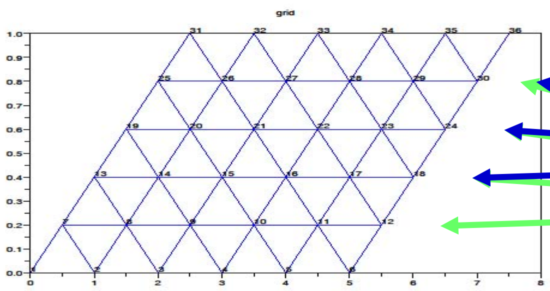
# Space-time meshes

# Well posed discrete system




Cartesian mesh:

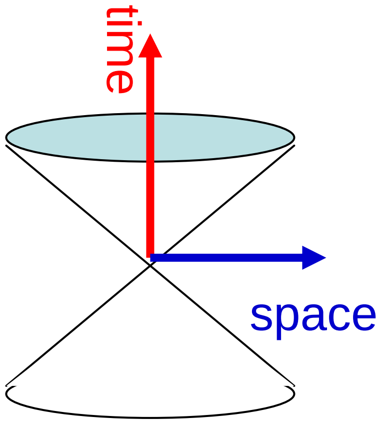
# ansatz functions
= # test functions



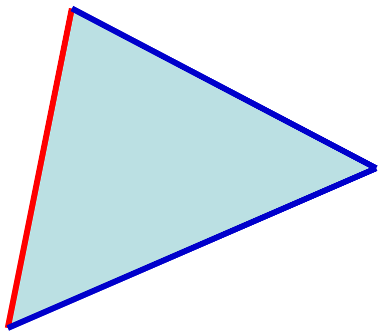

unstructured mesh:

# ansatz functions
= # test functions

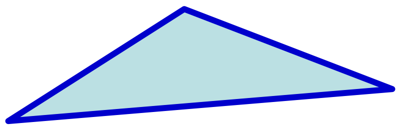
# CFL condition/ stability for unstructured meshes?




Cartesian mesh: $\Delta t < C \, \Delta x$

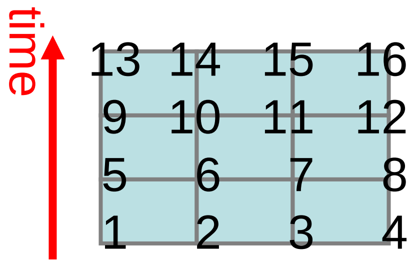


Simplex: one time edge
d space edges



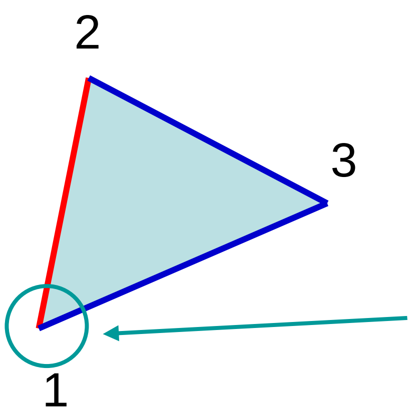
or: no time edge
d+1 space edges

# Explicit scheme for unstructured meshes?



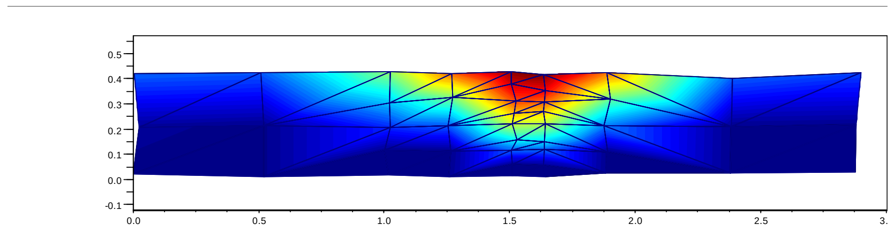

Cartesian mesh:
- nodes in ascending time order
- time slice by time slice

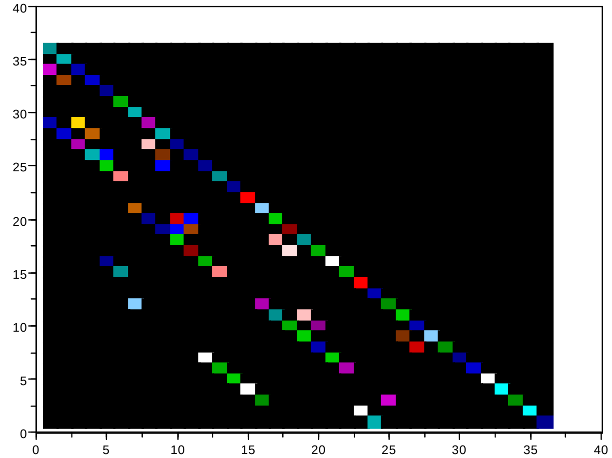
Uniform triangular grid:
- node by node



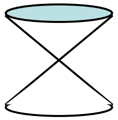

Simplex:
- node at begin of time edge first

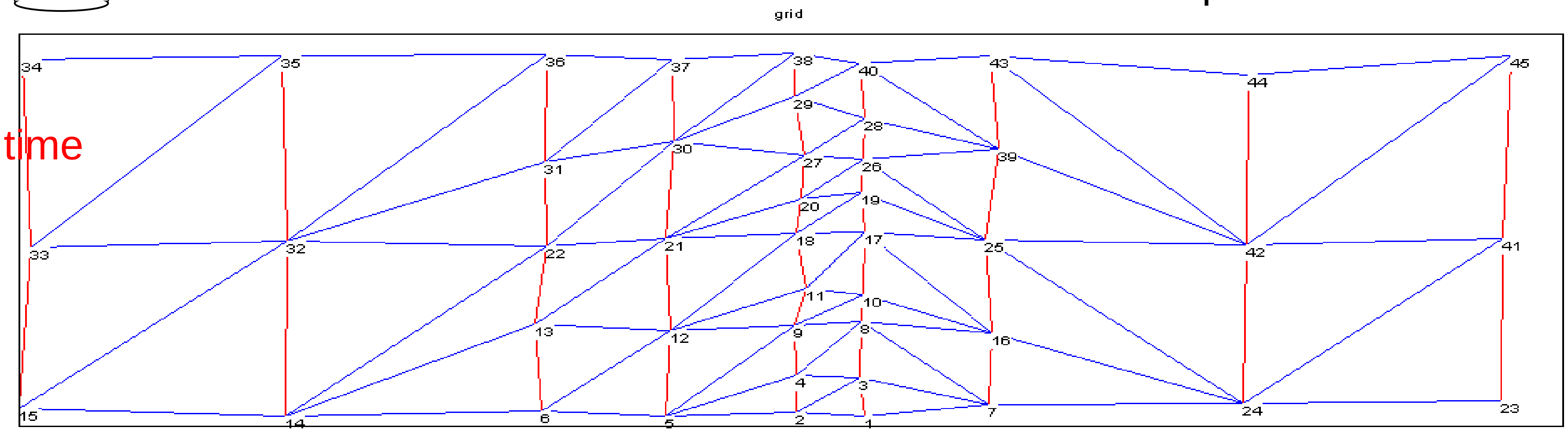
# Local time-stepping





explicit

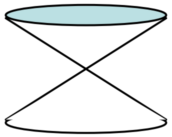


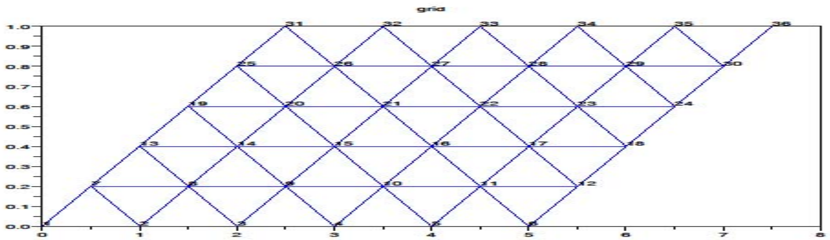


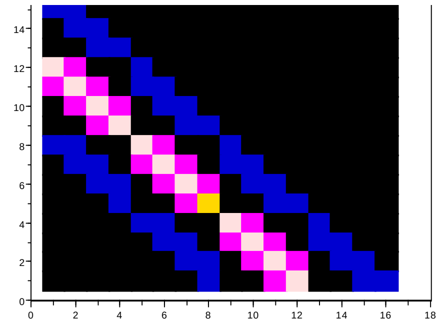

space

CLF with spatial mesh refinement+local time stepping

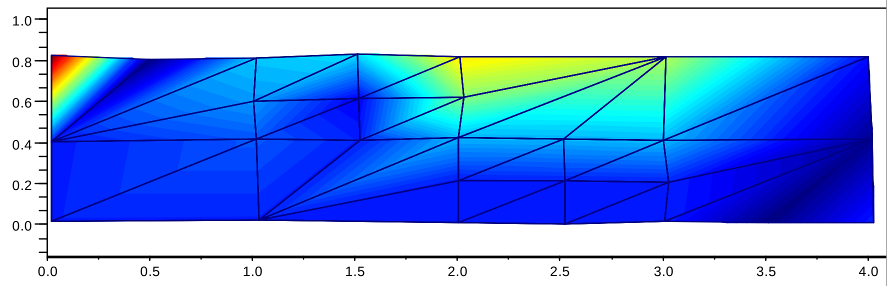
# No time edge

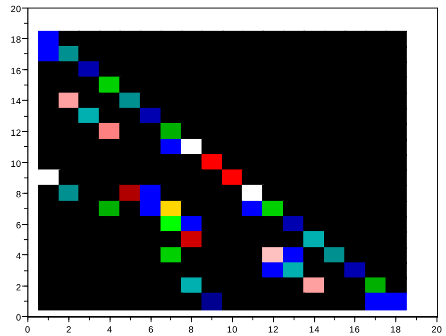




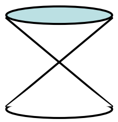



# Refined spacetime meshes

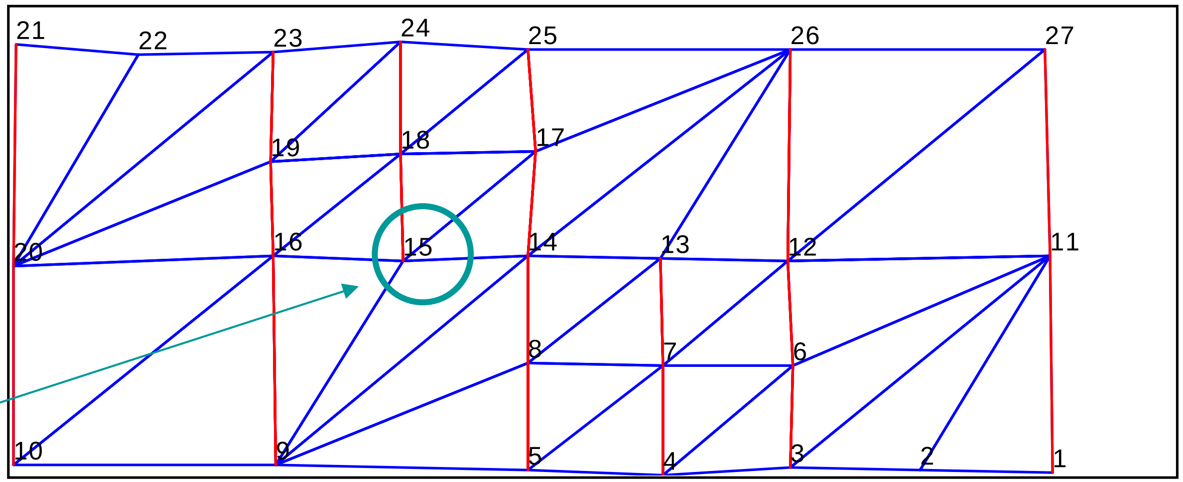



grid



time

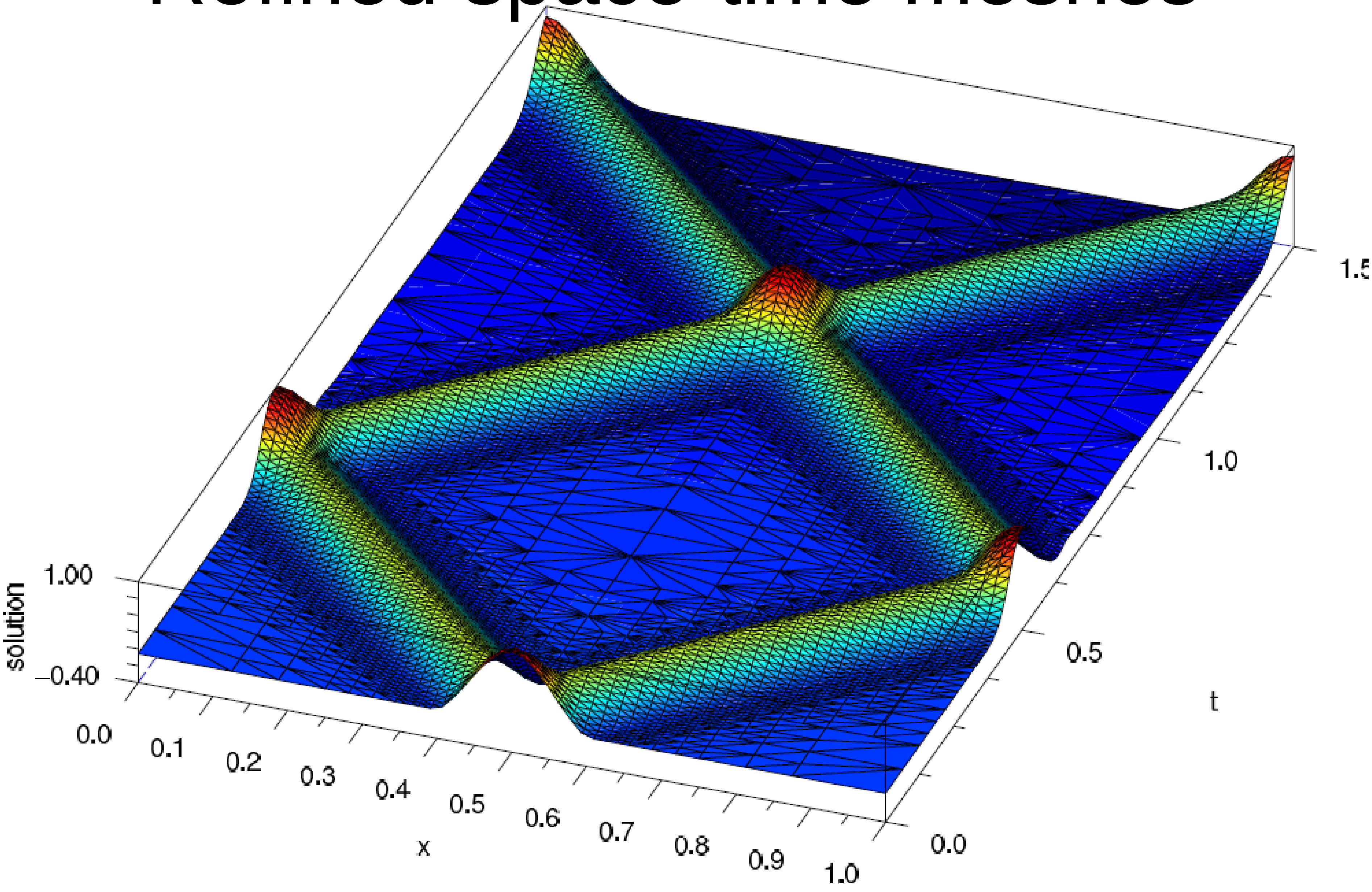



Interpolation/
hanging node

space

CLF with local space-time refinement

# Refined space-time meshes



# Einstein‘s equations

$$(g_{\alpha\beta}) = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}$$

metric

$$(g^{\alpha\beta}) = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}^{-1}$$

$$\Gamma^{\alpha}_{\beta\gamma} := \tfrac{1}{2} g^{\alpha\sigma}(\partial_\beta g_{\gamma\sigma} + \partial_\gamma g_{\beta\sigma} - \partial_\sigma g_{\beta\gamma})$$

connection

$$R_{\mu\nu} := \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\mu \Gamma^{\alpha}_{\nu\alpha} + \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}$$

$$R = g^{\alpha\beta} R_{\alpha\beta}$$

curvature

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Einstein equation

# Analogy: Maxwell equation

$$(A_\alpha) = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Vector potential

$$\partial^\alpha(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = 0$$

ill posed Cauchy problem

$$\partial^\alpha A_\alpha = 0$$

Lorenz gauge condition

$$\Box A_\alpha = 0$$

Maxwell equation in Lorenz gauge

$$dA = (F_{\alpha\beta}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$R_{\mu\nu} = 0$$

find metric $g_{\mu\nu}$ such that:
ill posed Cauchy problem

$$\Gamma^\alpha := g^{\rho\sigma}\Gamma^\alpha_{\rho\sigma}$$

$$\Gamma^\alpha = 0$$

harmonic gauge condition

$$R^{(h)}_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\alpha\nu}\partial_\mu \Gamma^\alpha - \frac{1}{2} g_{\alpha\mu}\partial_\nu \Gamma^\alpha$$

$$-\frac{1}{2} g^{\alpha\beta}\partial_\alpha \partial_\beta g_{\mu\nu}$$

principal part

$$R^{(h)}_{\mu\nu} = 0$$

in harmonic gauge:
symmetric hyperbolic
strong formulation

# weak formulation

$$S := \int_{\mathcal{M}} R\sqrt{-g}\,d^4x$$

Einstein-Hilbert action

$$\delta S = \int_{\mathcal{M}} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)(\delta g^{\mu\nu})\sqrt{-g}\,d^4x$$

$$g_{\alpha\beta} \in V_a \text{ such that } \boxed{\delta S = 0} \;\; \forall \delta g^{\mu\nu}$$

weak formulation
2nd derivatives

$$\int_{\mathcal{M}} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)(v^{\mu\nu}) \sqrt{-g} d^4x = 0$$

$$a(g,v) := \frac{1}{2} \int_{\mathcal{M}} g^{\alpha\beta} \sqrt{-g}\, (\partial_\alpha g_{\mu\nu})(\partial_\beta v^{\mu\nu}) d^4x$$

principal part

$$\begin{aligned} q(g,v) := \tfrac{1}{2} \textstyle\int_{\mathcal{M}} g^{\alpha\beta} g^{\rho\sigma} \sqrt{-g} \Big( & (\partial_\alpha g_{\rho\mu})(\partial_\beta g_{\sigma\nu}) - (\partial_\alpha g_{\rho\mu})(\partial_\sigma g_{\beta\nu}) \\ + & (\partial_\alpha g_{\rho\mu})(\partial_\nu g_{\beta\sigma}) + (\partial_\mu g_{\alpha\rho})(\partial_\beta g_{\sigma\nu}) \\ -\tfrac{1}{2} & (\partial_\mu g_{\alpha\rho})(\partial_\nu g_{\beta\sigma}) \Big) v^{\mu\nu} d^4x \ , \end{aligned}$$

$$g \in V_a \text{ such that } \begin{aligned} a(g,v) + q(g,v) &= 0 \\ \text{and } \Gamma^\alpha &= 0 \end{aligned} \ \forall v \in V_t .$$

weak
formulation
1st derivatives

$$b(g,v) := \int_{\mathcal{M}} (g_{\alpha\nu} g_{\mu\beta} - \frac{1}{2} g_{\mu\nu} g_{\alpha\beta}) \sqrt{-g}\, T^{\alpha\beta}\, v^{\mu\nu} d^4x$$

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Minkowski metric, flat background

$$\partial^\alpha = \eta^{\alpha\beta}\partial_\beta$$
$$\Box = \partial^\alpha\partial_\alpha$$
$$h_{\mu\nu} := g_{\mu\nu} - \tfrac{1}{2}\eta^{\alpha\beta}g_{\alpha\beta}$$

$$-\tfrac{1}{2}\Box h_{\mu\nu} = 0$$
$$\partial^\mu h_{\mu,\nu} = 0$$

linearized equation

$$a(h,v) := \frac{1}{2}\int_{\mathcal{M}} \eta^{\alpha\beta}(\partial_\alpha h_{\mu\nu})(\partial_\beta v^{\mu\nu})d^4x$$

seek $h \in V_a$ such that $a(h,v) = 0$ $\forall v \in V_t$

$$\partial^\mu h_{\mu,\nu} = 0$$

# nonlinear space-time FEM

$$a(g,v) := \frac{1}{2}\int_{\mathcal{M}} g^{\alpha\beta}\sqrt{-g}\,(\partial_\alpha g_{\mu\nu})(\partial_\beta v^{\mu\nu})d^4x$$

$$\begin{aligned} q(g,v) := \tfrac{1}{2}\textstyle\int_{\mathcal{M}} g^{\alpha\beta}g^{\rho\sigma}\sqrt{-g}\Big( & (\partial_\alpha g_{\rho\mu})(\partial_\beta g_{\sigma\nu}) - (\partial_\alpha g_{\rho\mu})(\partial_\sigma g_{\beta\nu}) \\ + & (\partial_\alpha g_{\rho\mu})(\partial_\nu g_{\beta\sigma}) + (\partial_\mu g_{\alpha\rho})(\partial_\beta g_{\sigma\nu}) \\ -\tfrac{1}{2} & (\partial_\mu g_{\alpha\rho})(\partial_\nu g_{\beta\sigma}) \Big) v^{\mu\nu} d^4x \;, \end{aligned}$$

$$\begin{aligned} g \in V_a \text{ such that } a(g,v) + q(g,v) &= 0 \;\forall v \in V_t \\ \text{and } \Gamma^\alpha &= 0 \;. \end{aligned}$$

$$\left(M - \frac{h_0^2}{6}A\right)u_{i+1} = \left(M + \frac{2h_0^2}{3}A\right)u_i - \left(M - \frac{h_0^2}{6}A\right)u_{i-1}$$

# nonlinear Symmetric Interior Penalty Discontinuous Galerkin

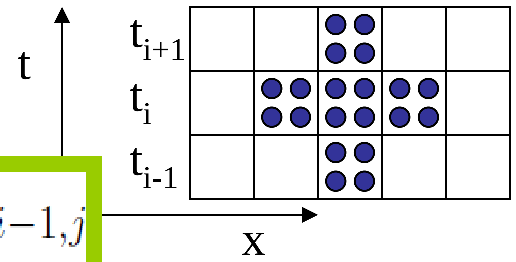
$$\{u\} := ((u|_{E_i}) + (u|_{E_j}))/2 \qquad [u] := (u|_{E_i}) - (u|_{E_j})$$

$$\begin{aligned} a(u,v) \quad := & \tfrac{1}{2}\sum_i \int_{E_i} g^{\alpha\beta}\sqrt{-g}\,(\partial_\alpha u)(\partial_\beta v)d^4x \\ & -\tfrac{1}{2}\sum_{i<j}\int_{e_{ij}}\{g^{\alpha\beta}\sqrt{-\hat{g}}\,n^{ij}_\alpha \partial_\beta u\}[v]d^3x \\ & -\tfrac{1}{2}\sum_{i<j}\int_{e_{ij}}[u]\{g^{\alpha\beta}\sqrt{-g}\,n^{ij}_\alpha \partial_\beta v\}d^3x \\ & +\tfrac{1}{2}\sum_{i<j}\frac{c_p}{|e_{ij}|^{ce}}\int_{e_{ij}}\{g^{\alpha\beta}n^{ij}_\alpha n^{ij}_\beta\sqrt{-g}\}[u][v]d^3x \end{aligned}$$

$$\begin{aligned} q(g,v) := \tfrac{1}{2}\int_{\mathcal{M}} g^{\alpha\beta}g^{\rho\sigma}\sqrt{-g}\Big( & (\partial_\alpha g_{\rho\mu})(\partial_\beta g_{\sigma\nu}) - (\partial_\alpha g_{\rho\mu})(\partial_\sigma g_{\beta\nu}) \\ + & (\partial_\alpha g_{\rho\mu})(\partial_\nu g_{\beta\sigma}) + (\partial_\mu g_{\alpha\rho})(\partial_\beta g_{\sigma\nu}) \\ -\tfrac{1}{2} & (\partial_\mu g_{\alpha\rho})(\partial_\nu g_{\beta\sigma}) \Big) v^{\mu\nu}d^4x \ , \end{aligned}$$

$$\begin{aligned} g \in V_a \text{ such that } a(g,v) + q(g,v) &= 0 \ \forall v \in V_t \\ \text{and } \Gamma^\alpha &= 0 \ . \end{aligned}$$

$$A_{1,0}u_{i+1,j} = A_{0,-1}u_{i,j-1} + A_{0,0}u_{i,j} + A_{0,1}u_{i,j+1} - A_{-1,0}u_{i-1,j}$$



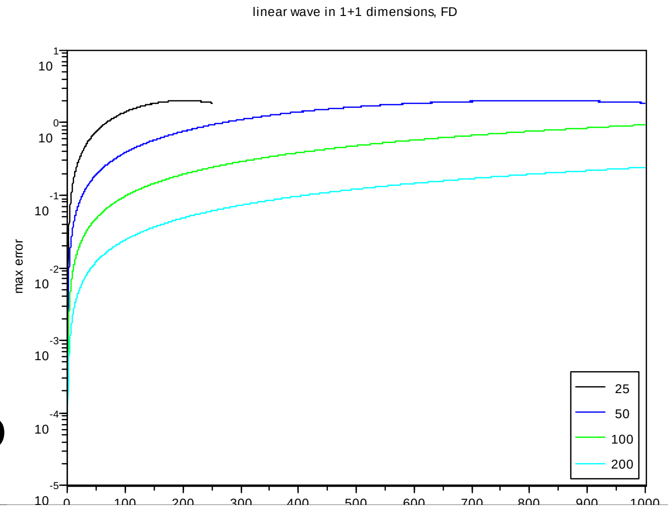
# Traveling wave 1+1

$$h_{00} = h_{11} = -h_{01} = \sin 2\pi(x_1 - x_0)$$

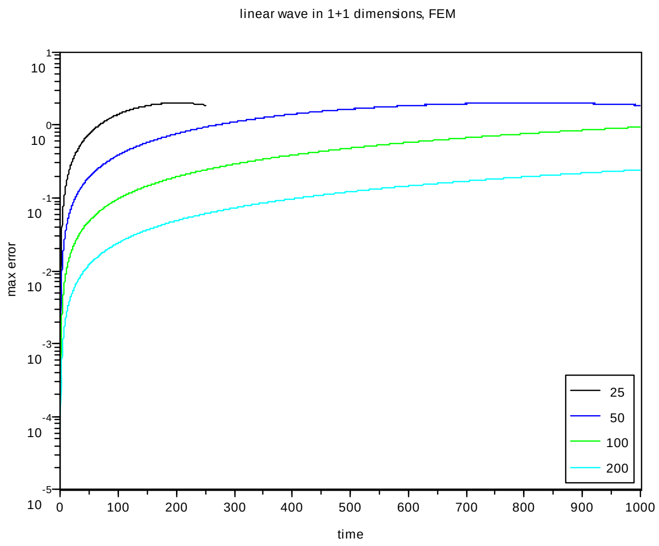
$$-\frac{1}{2}\Box h_{\mu\nu} = 0$$

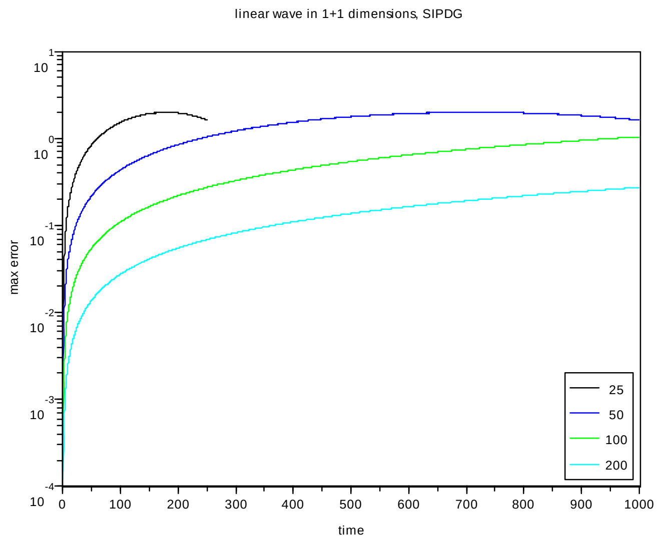
$$\partial^\mu h_{\mu,\nu} = 0$$

periodic b.c., domain [0,1]




FD



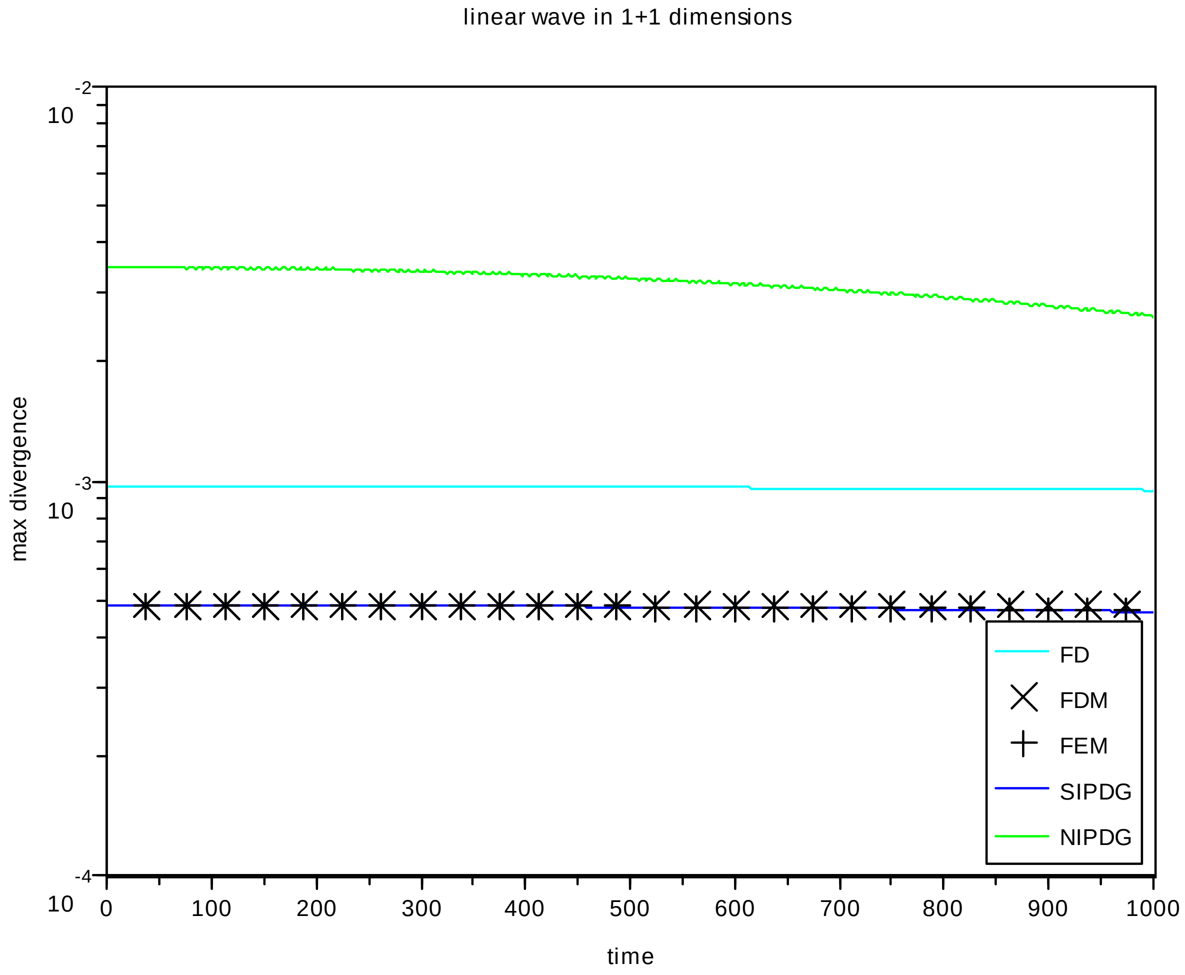




FEM

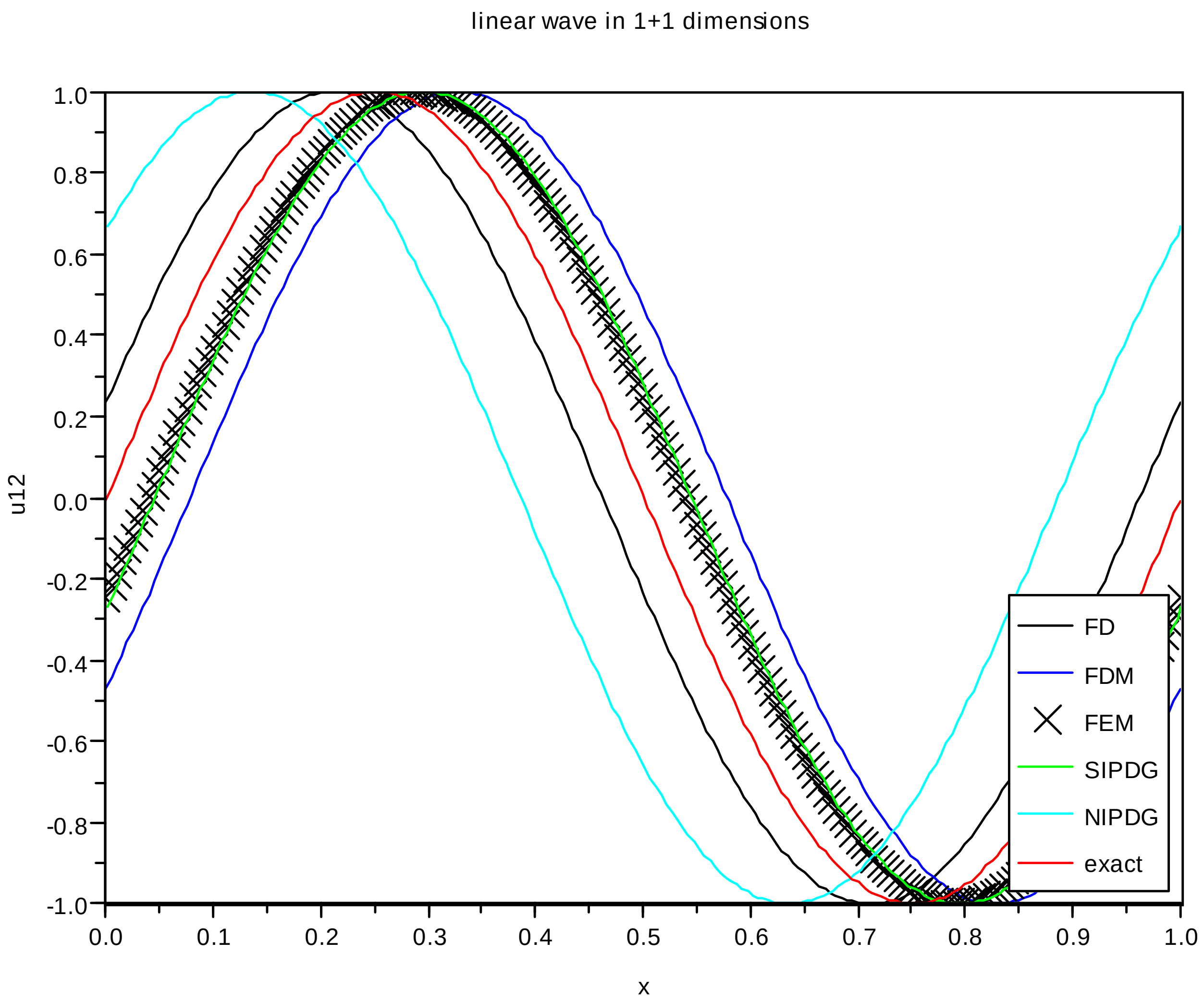
SIPDG

# gauge condition

$$\partial^\mu h_{\mu,\nu} = 0$$



# spatial error

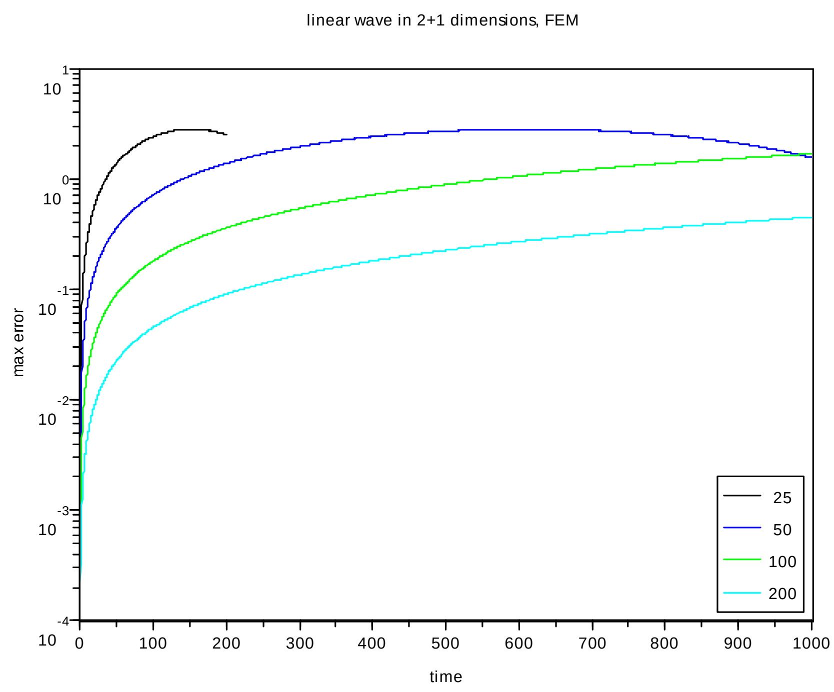


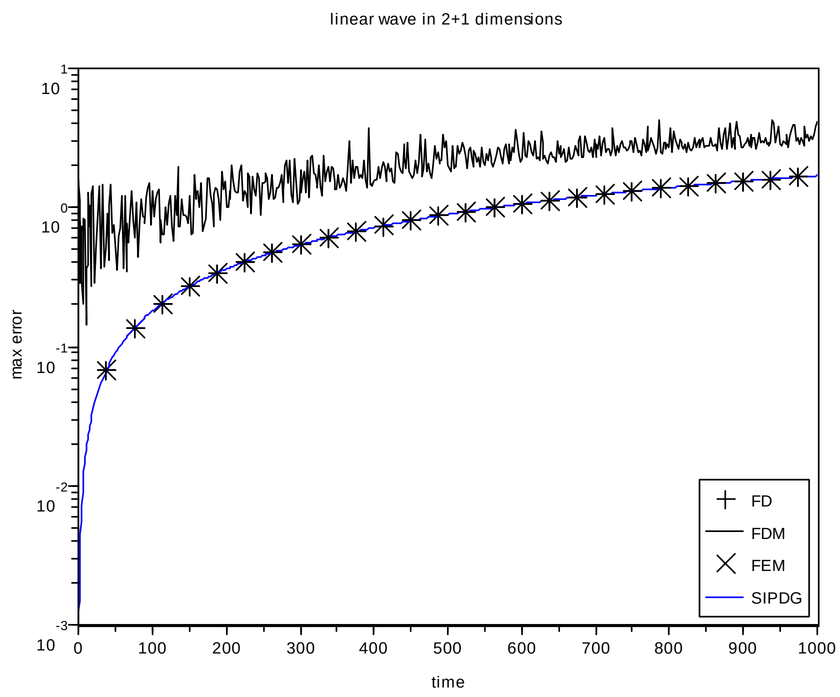
# Traveling wave 2+1

$$h_{11} = h_{22} = (\sqrt{2} - 1)h_{01} \qquad -\frac{1}{2}\Box h_{\mu\nu} = 0$$

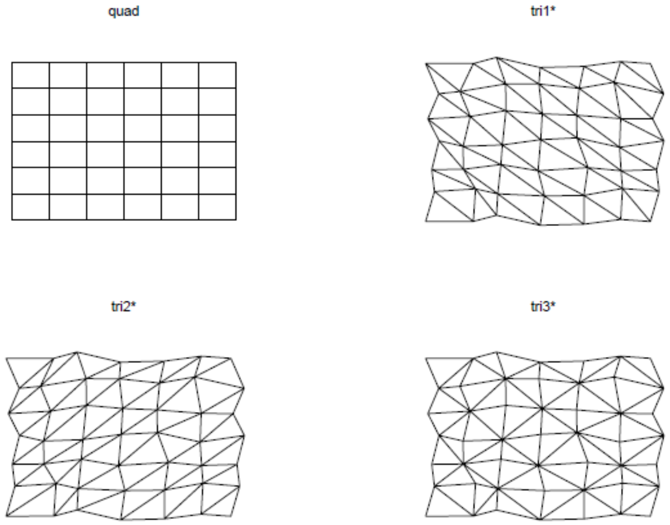
$$h_{01} = h_{02} = h_{12} = \sin 2\pi(x_1 + x_2 - \sqrt{2}x_2) \qquad \partial^{\mu} h_{\mu,\nu} = 0$$

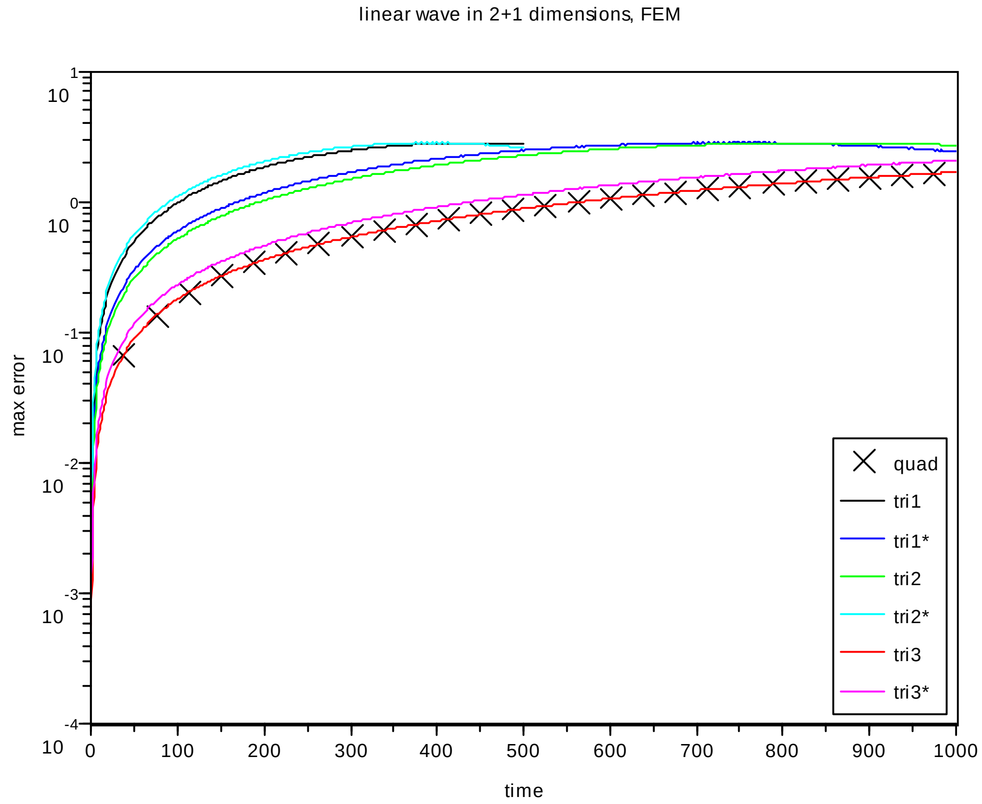
periodic b.c., domain $[0,1]^2$



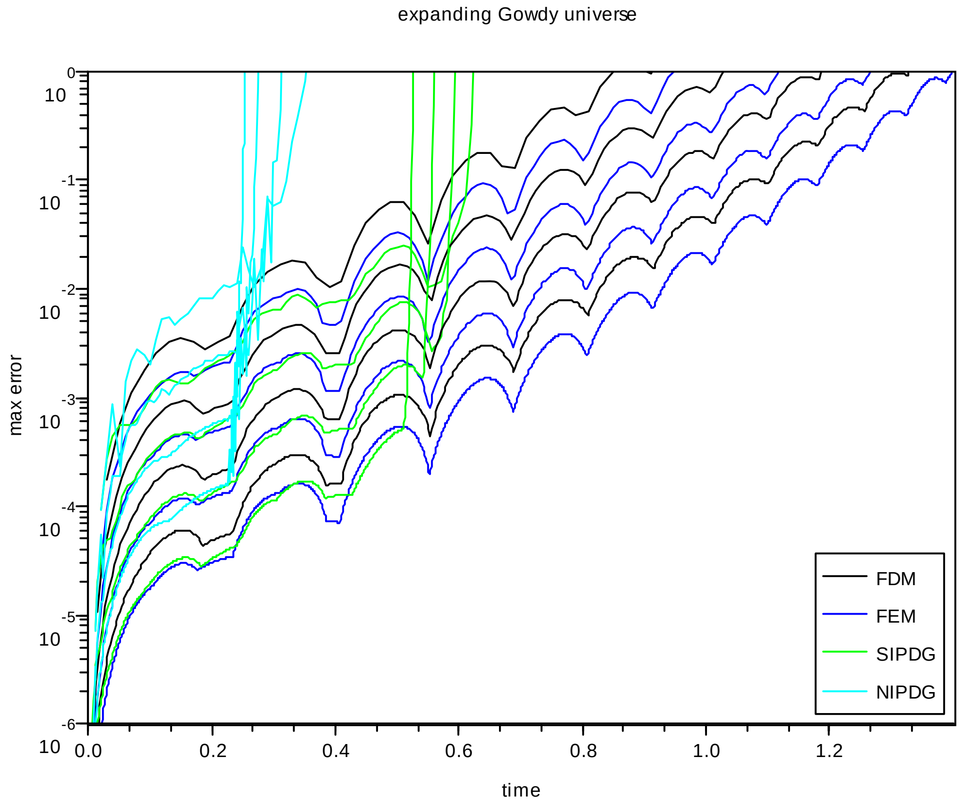
# quasi-uniform grid



# Gowdy universe

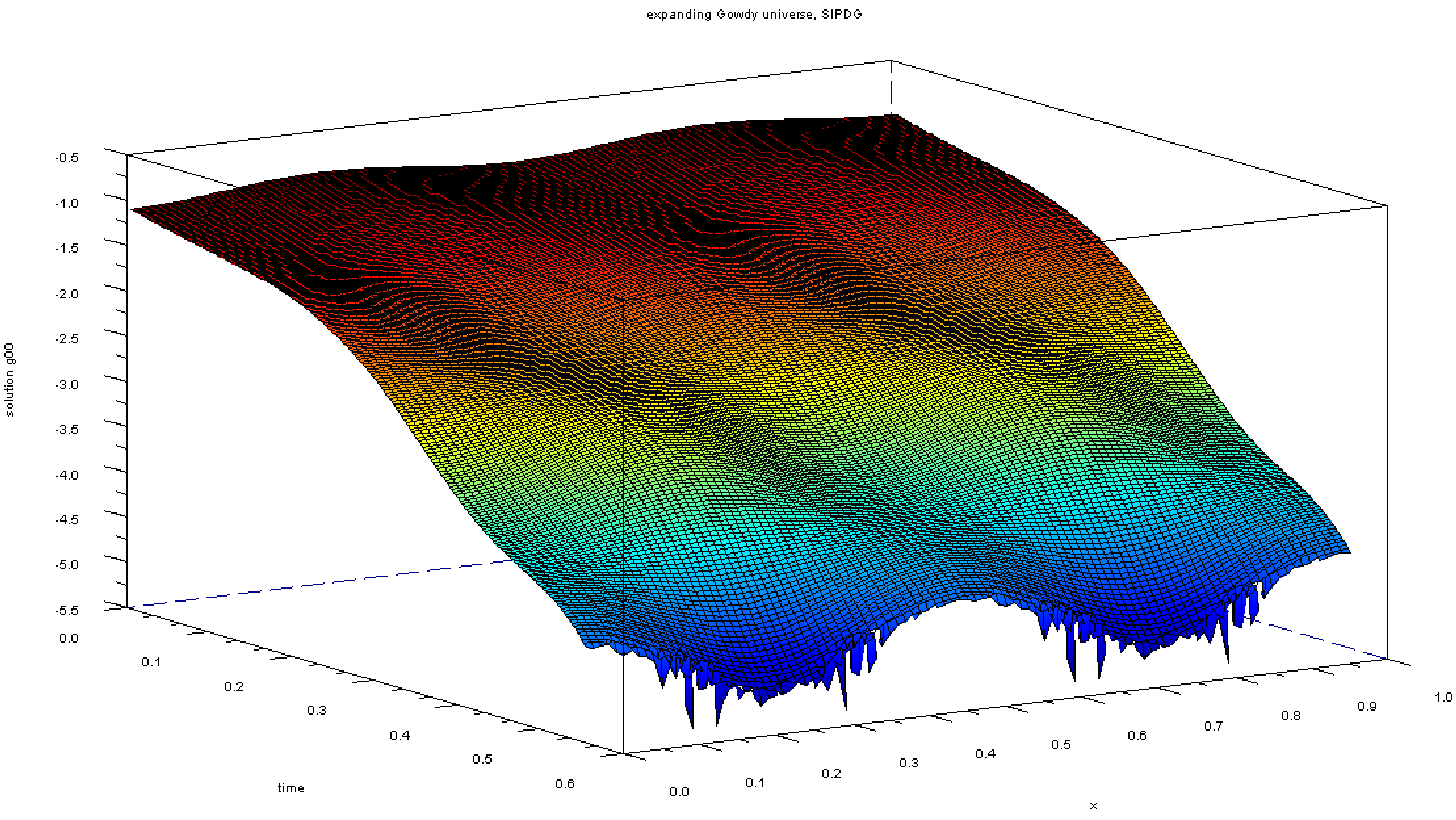
$$
\begin{aligned}
g &= \mathrm{diag}(-e^{(\lambda+3x_0)/2},\ e^{x_0+p},\ e^{x_0-p},\ e^{(\lambda-x_0)/2}) \text{ with} \\
p &:= \mathrm{J}_0(2\pi e^{x_0})\cos(2\pi x_3) \text{ and} \\
\lambda &:= -2\pi e^{x_0}\mathrm{J}_0(2\pi e^{x_0})\mathrm{J}_1(2\pi e^{x_0})\cos^2(2\pi x_3) - 2\pi \mathrm{J}_0(2\pi)\mathrm{J}_1(2\pi) \\
&\quad +2(\pi e^{x_0})^2(\mathrm{J}_0^2(2\pi e^{x_0}) + \mathrm{J}_1^2(2\pi e^{x_0})) - \tfrac{1}{2}(2\pi)^2(\mathrm{J}_0^2(2\pi) + \mathrm{J}_1^2(2\pi))
\end{aligned}
$$




polarized gravitational wave
expanding universe

start x0=1
periodic b.c., domain[0,1]
n=25,50,100,200

# Gowdy universe



# Conclusion

- Space-time FEM, DG for 2nd order wave eq.
- Unstructured space-time mesh criteria
- weak formulation, FEM, DG for gravity (Einstein-Hilbert action)